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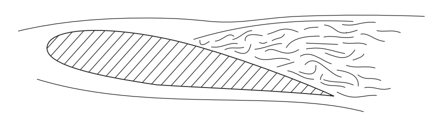

Flow of Ideal Fluids

7.1 INTRODUCTION

Flows at high Reynolds number reveal that the viscous effects are confined within the boundary layers. Far away from the solid surface, the flow is nearly inviscid and in many cases it is incompressible. We now aim at developing techniques for analyses of inviscid incompressible flows.

Incompressible flow is a constant density flow, and we assume ρ to be constant. We visualize a fluid element of defined mass moving along a streamline in an incompressible flow. Because the density is constant, we can write

$$\nabla \cdot \vec{\mathbf{V}} = 0 \tag{7.1}$$

Over and above, if the fluid element does not rotate as it moves along the streamline, or to be precise, if its motion is translational (and deformation with no rotation) only, the flow is termed as *irrotational flow*. It has already been shown in Sec. 3.3.5 that the motion of a fluid element can in general have translation, deformation and rotation. The rate of rotation of the fluid element can be measured by the average rate of rotation of two perpendicular line segments. The average rate of rotation ω_z about z-axis is expressed in terms of the gradients of velocity components (refer to Chapter 3) as

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Similarly, the other two components of rotation are

$$\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] \qquad \text{and} \qquad \omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]$$

As such, they are components of $\vec{\omega}$ which is given by

$$\vec{\omega} = \frac{1}{2}\left(\nabla \times \vec{V}\right)$$

In a two-dimensional flow, ω_z is the only non-trivial component of the rate of rotation. Imagine a pathline of a fluid particle shown in Fig. 7.1. Rate of spin of the particle is ω_z. The flow in which this spin is zero throughout is known as irrotational flow. A generalized statement is more appropriate: For irrotational flows, $\nabla \times \vec{\mathbf{V}} = 0$ in the flow field.

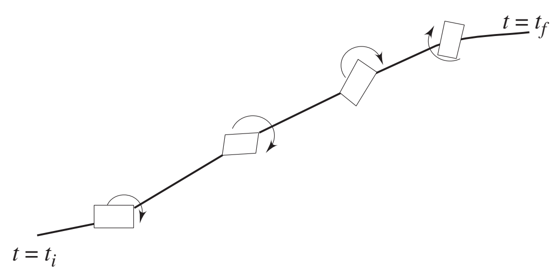

Fig. 7.1 Pathline of a fluid particle

Therefore for an irrotational flow, the velocity $\vec{\mathbf{V}}$ can be expressed as the gradient of a scalar function called the velocity potential, denoted by ϕ

$$\vec{\mathbf{V}} = \nabla\phi \tag{7.2}$$

Combination of Eqs (7.1) and (7.2) yields

$$\nabla^2 \phi = 0 \tag{7.3}$$

From Eq. (7.3) we see that an inviscid, incompressible, irrotational flow is governed by Laplace's equation.

Laplace's equation is linear, hence any number of particular solutions of Eq. (7.3) added together will yield another solution. This concept forms the building-block of the solution of inviscid, incompressible, irrotational flows. A complicated flow pattern for an inviscid, incompressible, irrotational flow can be synthesized by adding together a number of elementary flows which are also inviscid, incompressible and irrotational.

The analysis of Laplace's Eq. (7.3) and finding out the potential functions are known as *potential flow theory* and the inviscid, incompressible, irrotational flow is often called as potential flow. However, the following elementary flows can constitute several complex potential-flow problems

1. Uniform flow
2. Source or sink
3. Vortex

7.2 ELEMENTARY FLOWS IN A TWO-DIMENSIONAL PLANE

7.2.1 Uniform Flow

In this flow, velocity is uniform along y-axis and there exists only one component of velocity which is in the x direction. Magnitude of the velocity is U_0.
From Eq. (7.2) we can write

$$\hat{\mathbf{i}}\,U_0 + \hat{\mathbf{j}}\,0 = \hat{\mathbf{i}}\,\frac{\partial \phi}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial \phi}{\partial y}$$

or
$$\frac{\partial \phi}{\partial x} = U_0 \qquad \frac{\partial \phi}{\partial y} = 0$$

whence
$$\phi = U_0 x + C_1 \tag{7.4}$$

Recall from Sec. 4.2.2 that in a two dimensional flow field, flow can also be described by stream function ψ. In the case of uniform flow

$$\frac{\partial \psi}{\partial y} = U_0 \qquad \text{and} \qquad -\frac{\partial \psi}{\partial x} = 0$$

so that
$$\Psi = U_0 y + K_1 \tag{7.5}$$

The constants of integration C_1 and K_1 in Eqs (7.4) and (7.5) are arbitrary. The values of ψ and ϕ for different streamlines and velocity potential lines may change but flow pattern is unaltered. The constants of integration may be omitted and it is possible to write

$$\psi = U_0 y, \qquad \phi = U_0 x \tag{7.6}$$

These are plotted in Fig. 7.2(a) and consist of a rectangular mesh of straight streamlines and orthogonal straight potential-lines. It is conventional to put arrows on the streamlines showing the direction of flow.
In terms of polar ($r - \theta$) coordinate, Eq. (7.6) becomes

$$\psi = U_0\, r \sin \theta, \qquad \phi = U_0\, r \cos \theta \tag{7.7}$$

If we consider a uniform stream at an angle α to the x-axis as shown in Fig. 7.2b, we require that

$$u = U_0 \cos \alpha = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$$

and
$$v = U_0 \sin \alpha = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \tag{7.8}$$

Integrating, we obtain for a uniform velocity U_0 at an angle α, the stream function and velocity potential respectively as

$$\psi = U_0(y \cos\alpha - x \sin\alpha), \quad \phi = U_0\,(x \cos\alpha + y \sin\alpha) \tag{7.9}$$

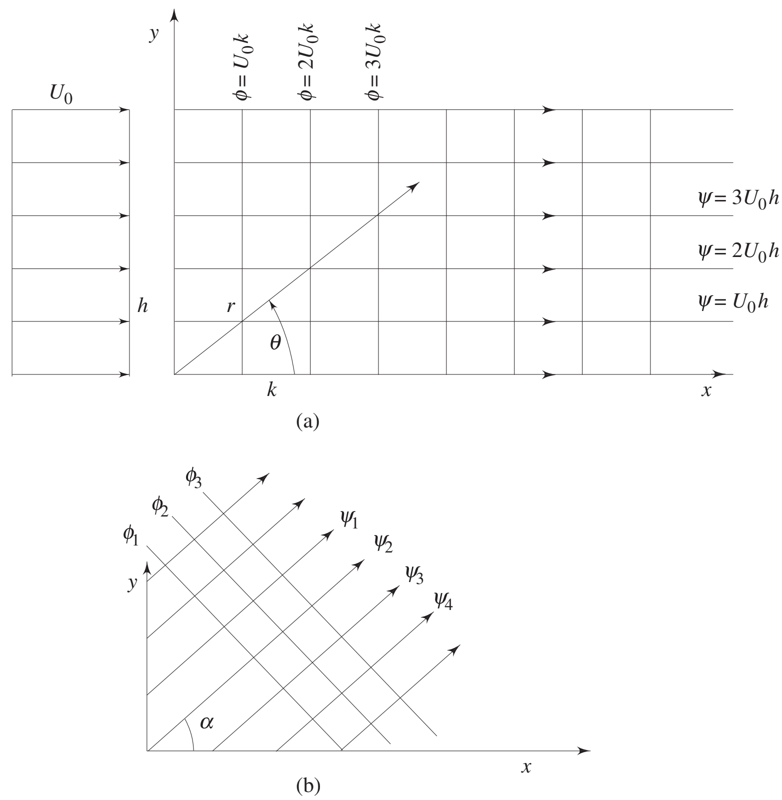

Fig. 7.2 (a) Flownet for a uniform stream
(b) Flownet for uniform stream with an angle α with x-axis

7.2.2 Source or Sink

Consider a flow with straight streamlines emerging from a point, where the velocity along each streamline varies inversely with distance from the point, as shown in Fig. 7.3. Only the radial component of velocity is non-trivial ($v_\theta = 0$, $v_z = 0$).

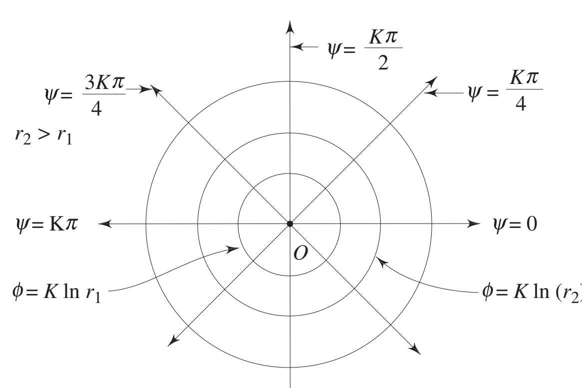

Fig. 7.3 Flownet for a source flow

Such a flow is called *source flow*. In a steady source flow the amount of fluid crossing any given cylindrical surface of radius r and unit length is constant ($\dot{m}$)

$$\dot{m} = 2\pi r\, v_r\, \rho$$

or
$$v_r = \frac{\dot{m}}{2\pi\rho}\cdot\frac{1}{r} = \frac{\Lambda}{2\pi}\cdot\frac{1}{r} = \frac{K}{r} \tag{7.10a}$$

where, K is the source strength

$$K = \frac{\dot{m}}{2\pi\rho} = \frac{\Lambda}{2\pi} \tag{7.10b}$$

and Λ is the volume flow rate

Again recall from Sec. 4.2.2 that the definition of stream function in cylindrical polar coordinate states that

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad v_\theta = -\frac{\partial\psi}{\partial r} \tag{7.11}$$

Now for the source flow, it can be said that

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{K}{r} \tag{7.12}$$

and
$$-\frac{\partial\psi}{\partial r} = 0 \tag{7.13}$$

Combining Eqs (7.12) and (7.13), we get

$$\psi = K\theta + C_1 \tag{7.14}$$

However, this flow is also irrotational and we can write

$$\hat{\mathbf{i}}\, v_r + \hat{\mathbf{j}}\, v_\theta = \hat{\mathbf{i}}\frac{\partial\phi}{\partial r} + \hat{\mathbf{j}}\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

or
$$v_r = \frac{\partial\phi}{\partial r} \quad \text{and} \quad v_\theta = 0 = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

or
$$\frac{\partial\phi}{\partial r} = v_r = \frac{K}{r} \quad \text{or} \quad \phi = K\ln r + C_2 \tag{7.15}$$

Likewise in uniform flow, the integration constants C_1 and C_2 in Eqs (7.14) and (7.15) have no effect on the basic structure of velocity and pressure in the flow. The equations for streamlines and velocity potential lines for source flow become

$$\psi = K\theta \quad \text{and} \quad \phi = K\ln r \tag{7.16}$$

where K is defined as the source strength and is proportional to Λ which is the rate of volume flow from the source per unit depth perpendicular to the page as shown in Fig. 7.3. If Λ is negative, we have sink flow, where the flow is in the opposite direction of the source flow. In Fig. 7.3, the point O is the origin of the radial streamlines. We visualize that point O is a point source or sink that induces radial flow in the neighbourhood. The point source or sink is a point of singularity in the flow field (because v_r becomes infinite). It can also be visualized that point O in Fig. 7.3 is simply a point formed by the intersection of plane of the paper and a

line perpendicular to the paper. The line perpendicular to the paper is a line source, with volume flow rate (Λ) per unit length. However, for sink, the stream function and velocity potential function are

$$\psi = -K\theta \quad \text{and} \quad \phi = -K \ln r \tag{7.17}$$

7.2.3 Vortex Flow

In this flow all the streamlines are concentric circles about a given point where the velocity along each streamline is inversely proportional to the distance from the centre, as shown in Fig. 7.4. Such a flow is called *vortex (free vortex) flow*. This flow is necessarily irrotational.

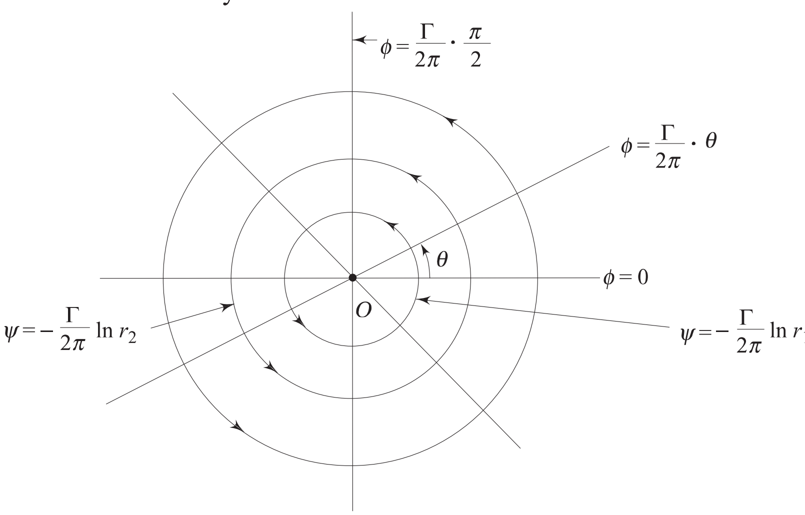

Fig. 7.4 Flownet for a vortex (free vortex)

In a *purely circulatory (free vortex flow) motion*, we can write the tangential velocity as

$$v_\theta = \frac{\text{Circulation constant}}{r}$$

$$v_\theta = \frac{\Gamma/2\pi}{r} \tag{7.18}$$

where Γ is circulation,

Also, for purely circulatory motion one can write

$$v_r = 0 \tag{7.19}$$

With the definition of stream function, it is evident that

$$v_\theta = -\frac{\partial \psi}{\partial r} \quad \text{and} \quad v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

Combining Eqs (7.18) and (7.19) with the above said relations for stream function, it is possible to write

$$\psi = -\frac{\Gamma}{2\pi} \ln r + C_1 \tag{7.20}$$

Because of irrotationality, it should satisfy

$$\hat{\mathbf{i}}\, v_r + \hat{\mathbf{j}}\, v_\theta = \hat{\mathbf{i}}\frac{\partial \phi}{\partial r} + \hat{\mathbf{j}}\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Eqs (7.18) and (7.19) and the above solution of Laplace's equation yields

$$\phi = \frac{\Gamma}{2\pi}\theta + C_2 \tag{7.21}$$

The integration constants C_1 and C_2 have no effect whatsoever on the structure of velocities or pressures in the flow. Therefore like other elementary flows, we shall consistently ignore such constants. It is clear that the *streamlines* for vortex flow are *circles* while the *potential* lines are *radial*. These are given by

$$\psi = -\frac{\Gamma}{2\pi}\ln r \qquad \text{and} \qquad \phi = \frac{\Gamma}{2\pi}\theta \tag{7.22}$$

In Fig. 7.4, point O can be imagined as a point vortex that induces the circulatory flow around it. The point vortex is a singularity in the flow field (v_θ becomes infinite). It is also discerned that the point O in Fig. 7.4 is simply a point formed by the intersection of the plane of a paper and a line perpendicular to the plane. This line is called *vortex filament* of strength Γ, where Γ is the circulation around the *vortex filament* and the circulation is defined as

$$\Gamma = \oint \vec{\mathbf{V}} \cdot \mathbf{d}\vec{\mathbf{s}} \tag{7.23}$$

In Eq. (7.23), the line integral of the velocity component tangent to a curve of elemental length ds, is taken around a closed curve. It may be stated that the circulation for a closed path in an irrotational flow field is zero. However, the circulation for a given path in an irrotational flow containing a finite number of singular points is constant. In general this circulation constant Γ denotes the algebraic strength of the vortex filament contained within the closed curve.
From Eq. (7.23) we can write

$$\Gamma = \oint \vec{\mathbf{V}} \cdot \mathbf{d}\vec{\mathbf{s}} = \oint (u\mathrm{d}x + v\mathrm{d}y + w\mathrm{d}z)$$

For a two-dimensional flow

$$\Gamma = \oint (u\mathrm{d}x + v\,\mathrm{d}y)$$

or

$$\boldsymbol{\Gamma} = \oint V\cos\alpha\ \mathrm{d}s \tag{7.24}$$

Consider a fluid element as shown in Fig. 7.5. Circulation is positive in the anti-clockwise direction (not a mandatory but general convention).

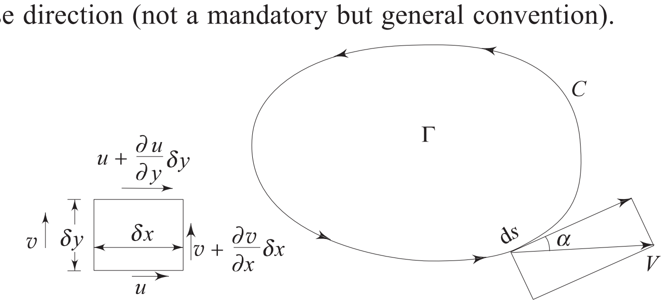

Fig. 7.5 Circulation in a flow field

$$\delta \Gamma = u\,\delta x + \left(v + \frac{\partial v}{\partial x}\delta x\right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y\right)\delta x - v\delta y$$

or $$\delta \Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta x\,\delta y$$

or $$\delta \Gamma = 2\,\omega_z\,\delta A$$

or $$\delta \Gamma/\delta A = 2\omega_z = \Omega_z \tag{7.25}$$

Physically, circulation per unit area is the vorticity of the flow.

Now, for a free vortex flow, the tangential velocity is given by Eq. (7.18) as

$$v_\theta = \frac{\Gamma/2\pi}{r} = \frac{C}{r}$$

For a circular path (refer Fig. 7.5)

$$\alpha = 0, \qquad V = v_\theta = \frac{C}{r}$$

Thus, $$\Gamma = \oint_0^{2\pi} \frac{C}{r}\, r\,\mathrm{d}\theta = 2\pi C \tag{7.26}$$

It may be noted that although free vortex is basically an irrotational motion, the circulation for a given path containing a singular point (including the origin) is constant ($2\pi C$) and independent of the radius of a circular streamline. However, if the circulation is calculated in a free vortex flow along any closed contour excluding the singular point (the origin), it should be zero. Let us look at Fig. 7.6 (a) and take a closed contour $ABCD$ in order to find out circulation about the point, P around $ABCD$

$$\Gamma_{ABCD} = -v_{\theta_{AB}} r_1\,\mathrm{d}\theta - v_{r_{BC}}(r_2 - r_1) + v_{\theta_{CD}} r_2\,\mathrm{d}\theta + v_{r_{DA}}(r_2 - r_1)$$

There is no radial flow

$$v_{r_{BC}} = v_{r_{DA}} = 0,\ v_{\theta_{AB}} = \frac{C}{r_1} \text{ and } v_{\theta_{CD}} = \frac{C}{r_2}$$

$$\Gamma_{ABCD} = \frac{-C}{r_1}\cdot r_1\mathrm{d}\theta + \frac{C}{r_2}\cdot r_2\mathrm{d}\theta = 0 \tag{7.27}$$

If there exists a solid body rotation at constant ω induced by some external mechanism, the flow should be called a *forced vortex motion* (Fig. 7.6b) and we can write

$$v_\theta = \omega r \qquad \text{and}$$

$$\Gamma = \oint v_\theta\,\mathrm{d}s = \oint_0^{2\pi} \omega r\cdot r\mathrm{d}\theta = 2\pi r^2\omega \tag{7.28}$$

Equation (7.28) predicts that the circulation is zero at the origin and it increases with increasing radius. The variation is parabolic.

It may be mentioned that the free vortex (irrotational) flow at the origin (Fig. 7.6a) is impossible because of mathematical singularity. However,

physically there should exist a rotational (forced vortex) core which is shown by the dotted line. Below are given two statements which are related to Kelvin's circulation theorem (stated in 1869) and Cauchy's theorem on irrotational motion (stated in 1815) respectively

(i) The circulation around any closed contour is invariant with time in an inviscid fluid.

(ii) A body of inviscid fluid in irrotational motion continues to move irrotationally.

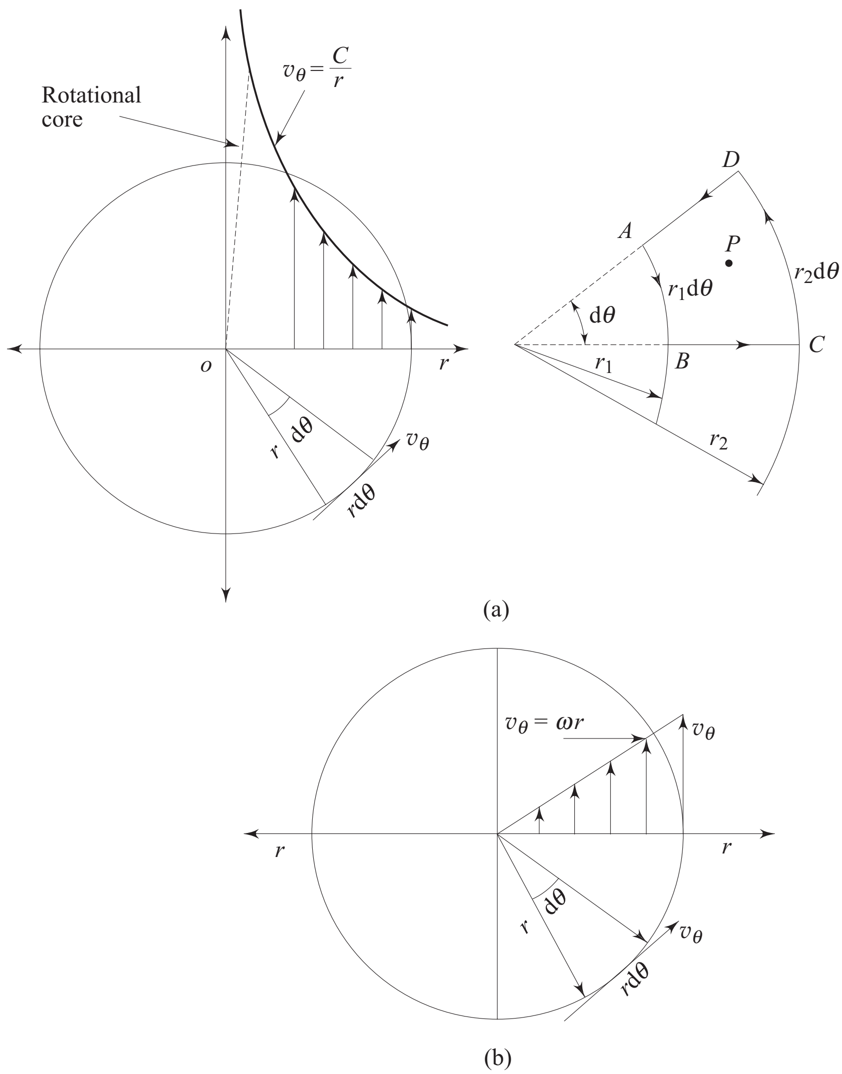

Fig. 7.6 (a) Free vortex flow (b) Forced vortex flow

7.3 SUPERPOSITION OF ELEMENTARY FLOWS

We can now form different flow patterns by superimposing the velocity potential and stream functions of the elementary flows stated above.

7.3.1 Doublet

In order to develop a doublet, imagine a source and a sink of equal strength K at equal distance s from the origin along x-axis as shown in Fig. 7.7.

From any point $P(x, y)$ in the field, r_1 and r_2 are drawn to the source and the sink. The polar coordinates of this point (r, θ) have been shown.

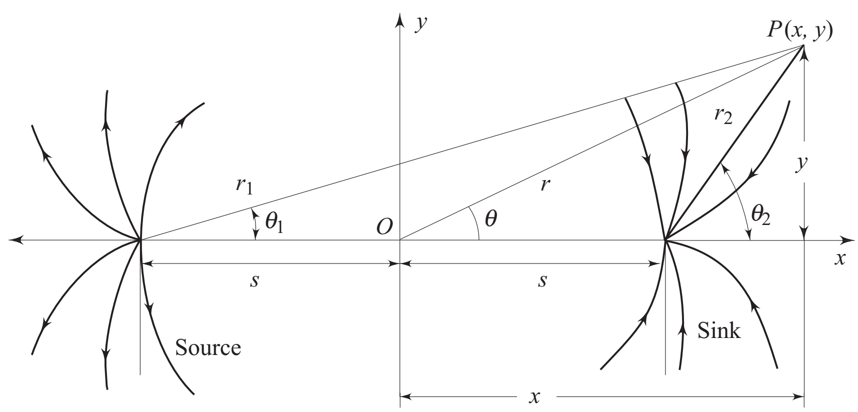

Fig. 7.7 Superposition of a source and a sink

The potential functions of the two flows may be superimposed to describe the potential for the combined flow at P as

$$\phi = K \ln r_1 - K \ln r_2 \tag{7.29}$$

Similarly

$$\psi = K(\theta_1 - \theta_2) = -K\alpha \tag{7.30}$$

where, $\alpha = (\theta_2 - \theta_1)$

We can also write

$$\tan\theta_1 = \frac{y}{x+s} \quad \text{and} \quad \tan\theta_2 = \frac{y}{x-s} \tag{7.31}$$

$$r_1 = \sqrt{r^2 + s^2 + 2rs\cos\theta} \quad \text{and} \quad r_2 = \sqrt{r^2 + s^2 - 2rs\cos\theta} \tag{7.32}$$

Now using the above mentioned relations we find

$$\tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

or

$$\tan\alpha = \left[\frac{yx + ys - yx + ys}{x^2 - s^2}\right] \Bigg/ \left(1 + \frac{y^2}{x^2 - s^2}\right)$$

or $$\tan\,\alpha = \frac{2ys}{x^2 + y^2 - s^2} \tag{7.33}$$

Hence the stream function and the velocity potential function are formed by combining Eqs (7.30) and (7.33), as well as Eqs (7.29) and (7.32) respectively

$$\psi = -K\tan^{-1}\left(\frac{2ys}{x^2 + y^2 - s^2}\right) \tag{7.34}$$

$$\phi = \frac{K}{2}\ln\left(\frac{r^2 + s^2 + 2rs\cos\theta}{r^2 + s^2 - 2rs\cos\theta}\right) \tag{7.35}$$

Doublet is a special case when a source as well as a sink are brought together in such a way that $s \to 0$ and at the same time the strength K ($\Lambda/2\pi$) is increased to an infinite value. These are assumed to be accomplished in a manner which makes the product of s and $\dfrac{\Lambda}{\pi}$ (in limiting case) a finite value χ. Under the aforesaid circumstances

$$\psi = -\frac{\Lambda}{2\pi}\cdot\frac{2ys}{x^2 + y^2 - s^2}$$

[Since in the limiting case $\tan^{-1}\alpha = \alpha$]

$$\psi = -\chi\cdot\frac{y}{x^2 + y^2} = \frac{-\chi\sin\theta}{r} \tag{7.36}$$

From Eq. (7.35), we get

$$\phi = \frac{\Lambda}{4\pi}\left[\ln\left(r^2 + s^2 + 2rs\cos\theta\right) - \ln\left(r^2 + s^2 - 2rs\cos\theta\right)\right]$$

or $$\phi = \frac{\Lambda}{4\pi}\left[\ln\left\{\left(r^2 + s^2\right)\left(1 + \frac{2rs\cos\theta}{r^2 + s^2}\right)\right\} - \ln\left\{\left(r^2 + s^2\right)\left(1 - \frac{2rs\cos\theta}{r^2 + s^2}\right)\right\}\right]$$

or $$\phi = \frac{\Lambda}{4\pi}\left[\left\{\frac{2rs\cos}{r^2 + s^2} - \frac{1}{2}\left(\frac{2rs\cos\theta}{r^2 + s^2}\right)^2 + \frac{1}{3}\left[\frac{2rs\cos\theta}{r^2 + s^2}\right]^3 + \cdots\right\}\right.$$
$$\left. - \left\{-\frac{2rs\cos\theta}{r^2 + s^2} - \frac{1}{2}\left(\frac{2rs\cos\theta}{r^2 + s^2}\right)^2 - \frac{1}{3}\left[\frac{2rs\cos\theta}{r^2 + s^2}\right]^3 + \cdots\right\}\right]$$

or $$\phi = \frac{\Lambda}{4\pi}\left[\frac{4rs\cos\theta}{r^2 + s^2} + \frac{2}{3}\left(\frac{2rs\cos\theta}{r^2 + s^2}\right)^3 + \cdots\right]$$

In the limiting condition the above expression can be written as

$$\phi \approx \frac{\chi r\cos\theta}{r^2 + s^2}$$

or
$$\phi \approx \frac{\chi \cos\theta}{r} \tag{7.37}$$

We can see that the streamlines associated with the doublet are
$$-\frac{\chi \sin\theta}{r} = C_1$$

If we replace $\sin\theta$ by y/r, and the minus sign be absorbed in C_1, we get
$$\chi \frac{y}{r^2} = C_1 \tag{7.38a}$$

In terms of cartesian coordinate, it is possible to write
$$x^2 + y^2 - \frac{\chi}{C_1} y = 0 \tag{7.38b}$$

Equation (7.38b) represents a family of circles. For $x = 0$, there are two values of y, one of which is zero. The centres of the circles fall on the y-axis. On the circle, where $y = 0$, x has to be zero for all the values of the constant. It is obvious that the family of circles formed due to different values of C_1 must be tangent to x-axis at the origin. These streamlines are illustrated in Fig. 7.8. Due to the initial positions of the source and the sink in the development of the doublet, it is certain that the flow will emerge in the negative x direction from the origin and it will converge via the positive x direction of the origin.

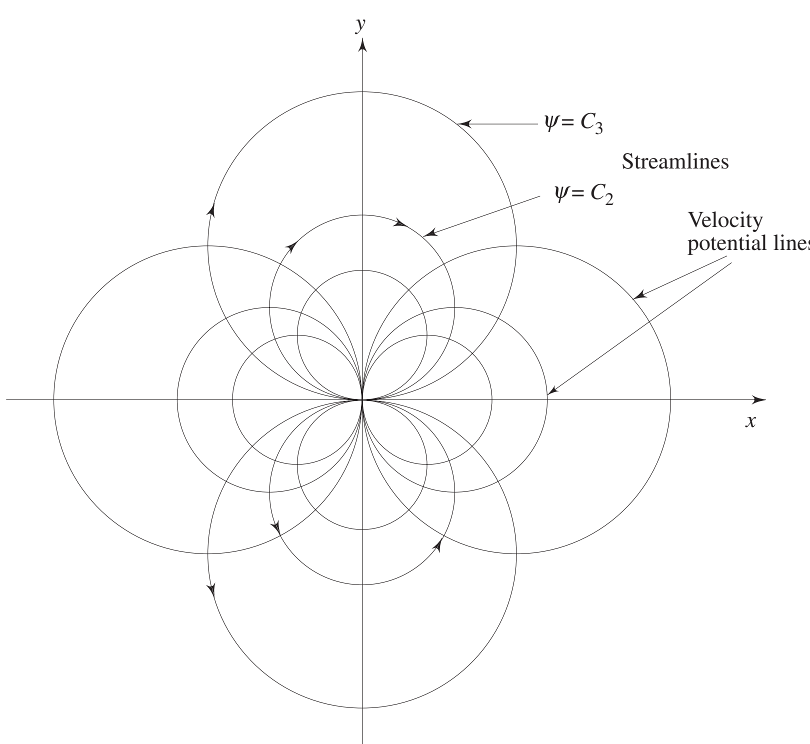

Fig. 7.8 Streamlines and velocity potential lines for a doublet

However, the velocity potential lines are

$$\frac{\chi \cos\theta}{r} = K_1$$

In cartesian coordinate this equation becomes

$$x^2 + y^2 - \frac{\chi}{K_1} x = 0 \qquad (7.39)$$

Once again we shall obtain a family of circles. The centres will fall on x-axis. For $y = 0$ there are two values of x, one of which is zero. When $x = 0$, y has to be zero for all values of the constant. Therefore these circles are tangent to y-axis at the origin. The orthogonality of constant ψ and constant ϕ lines are maintained as we iron out the procedure of drawing constant value lines (Fig. 7.8). In addition to the determination of the stream function and velocity potential, it is observed from Eq. (7.37) that for a doublet

$$v_r = \frac{\partial \phi}{\partial r} = \frac{-\chi \cos\theta}{r^2} \qquad (7.40)$$

As the centre of the doublet is approached, the radial velocity tends to be infinite. It shows that the doublet flow has a singularity. Since the circulation about a singular point of a source or a sink is zero for any strength, it is obvious that the circulation about the singular point in a doublet flow must be zero. It follows that for all paths in a doublet flow $\Gamma = 0$

$$\Gamma = \oint \vec{\mathbf{V}} \cdot \mathbf{d}\vec{\mathbf{s}} = 0 \qquad (7.41)$$

Applying Stokes Theorem between the line integral and the area-integral

$$\Gamma = \iint (\nabla \times \vec{\mathbf{V}})\, \mathrm{d}\vec{A} = 0 \qquad (7.42)$$

From Eq. (7.42), the obvious conclusion is $\nabla \times \vec{\mathbf{V}} = 0$, i.e., doublet flow is an irrotational flow.

At large distances from a doublet, the flow approximates the disturbances of a two dimensional airfoil. The influence of an airfoil as felt at distant walls may be approximated mathematically by a combination of doublets with varying strengths. Thus the cruise conditions of a two dimensional airfoil can be simulated by the superposition of a uniform flow and a doublet sheet of varying strengths.

7.3.2 Flow About a Cylinder Without Circulation

Inviscid-incompressible flow about a cylinder in uniform flow is equivalent to the superposition of a uniform flow and a doublet. The doublet has its axis of development parallel to the direction of the uniform flow. The combined potential of this flow is given by

$$\phi = U_0 x + \frac{\chi \cos\theta}{r} \qquad (7.43)$$

and consequently the stream function becomes

$$\psi = U_0 y - \frac{\chi \sin\theta}{r} \tag{7.44}$$

In our analysis, we shall draw streamlines in the flow field. Intwo-dimensional flow, a streamline may be interpreted as the edge of a surface on which the velocity vector should always be tangent and there is no flow in the direction normal to it. The latter is identically the characteristics of a solid impervious boundary. Hence, a streamline may also be considered as the contour of an *impervious two-dimensional body*. Figure 7.9 shows a set of streamlines. The streamline *C-D* may be considered as the edge of a two-dimensional body while the remaining streamlines form the flow about the boundary.

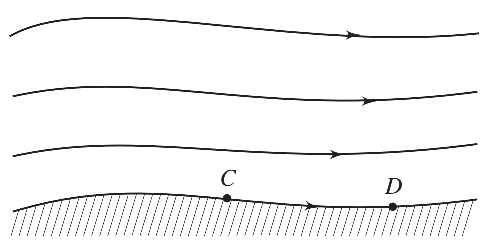

Fig. 7.9 Surface streamline

Now we follow the essential steps involving the superposition of elementary flows in order to form a flow about the body of interest. A streamline has to be determined which encloses an area whose shape is of practical importance in fluid flow. This streamline will describe the boundary of a two-dimensional solid body. The remaining streamlines outside this solid region will constitute the flow about this body.

Let us look for the streamline whose value is zero. Thus we obtain

$$U_0 y - \frac{\chi \sin\theta}{r} = 0 \tag{7.45}$$

replacing y by $r \sin\theta$, we have

$$\sin\theta \left(U_0 r - \frac{\chi}{r} \right) = 0 \tag{7.46}$$

If $\theta = 0$ or $\theta = \pi$, the equation is satisfied. This indicates that the x-axis is a part of the streamline $\psi = 0$. When the quantity in the parentheses is zero, the equation is identically satisfied. Hence it follows that

$$r = \left(\frac{\chi}{U_0} \right)^{1/2} \tag{7.47}$$

It can be said that there is a circle of radius $\left(\frac{\chi}{U_0} \right)^{1/2}$ which is an intrinsic part of the streamline $\psi = 0$. This is shown in Fig. 7.10. Let us look at the points of

intersection of the circle and x-axis, i.e. the points A and B. The polar coordinates of these points are

$$r = \left(\frac{\chi}{U_0}\right)^{1/2}, \qquad \theta = \pi, \text{ for point A}$$

$$r = \left(\frac{\chi}{U_0}\right)^{1/2}, \qquad \theta = 0, \text{ for point B}$$

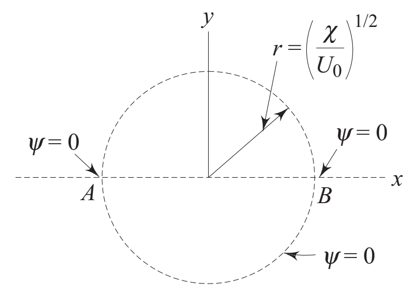

Fig. 7.10 Streamline $\psi = 0$ in a superimposed flow of doublet and uniform stream

The velocity at these points are found out by taking partial derivatives of the velocity potential in two orthogonal directions and then substituting the proper values of the coordinates. Thus

$$v_r = \frac{\partial \phi}{\partial r} = U_0 \cos\theta - \frac{\chi \cos\theta}{r^2} \tag{7.48a}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U_0 \sin\theta - \frac{\chi \sin\theta}{r^2} \tag{7.48b}$$

At point A $\left[\theta = \pi, \quad r = \left(\frac{\chi}{U_0}\right)^{1/2}\right]$

$$v_r = 0, \; v_\theta = 0$$

At point B $\left[\theta = 0, \quad r = \left(\frac{\chi}{U_0}\right)^{1/2}\right]$

$$v_r = 0, \; v_\theta = 0$$

The points A and B are clearly the stagnation points through which the flow divides and subsequently reunites forming a zone of circular bluff body.

The circular region, enclosed by part of the streamline $\psi = 0$ could be imagined as a solid cylinder in an inviscid flow. At a large distance from the cylinder the flow is moving uniformly in a cross-flow configuration.

Figure 7.11 shows the streamlines of the flow. The streamlines outside the circle describe the flow pattern of the inviscid irrotational flow across a cylinder. However, the streamlines inside the circle may be disregarded since this region is considered as a solid obstacle.

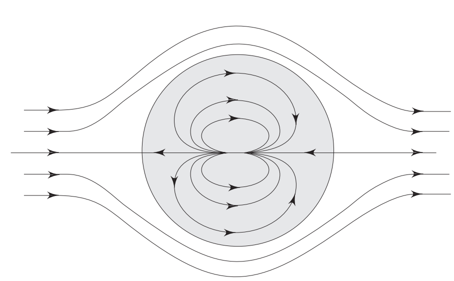

Fig. 7.11 Inviscid flow past a cylinder

7.3.3 Lift and Drag for Flow Past a Cylinder Without Circulation

Lift and drag are the forces per unit length on the cylinder in the directions normal and parallel respectively, to the direction of uniform flow.

Pressure for the combined doublet and uniform flow becomes uniform at large distances from the cylinder where the influence of doublet is indeed small. Let us imagine the pressure p_0 is known as well as uniform velocity U_0. Now we can apply Bernoulli's equation between infinity and the points on the boundary of the cylinder. Neglecting the variation of potential energy-between the aforesaid point at infinity and any point on the surface of the cylinder, we can write

$$\frac{p_0}{\rho g} + \frac{U_0^2}{2g} = \frac{p_b}{\rho g} + \frac{U_b^2}{2g} \tag{7.49}$$

where, the subscript b indicates the surface on the cylinder. As we know, since fluid cannot penetrate the solid boundary, the velocity U_b should be only in the transverse direction, or in other words, only v_θ component of velocity is present on the streamline $\psi = 0$.

Thus at $\quad r = \left(\dfrac{\chi}{U_0}\right)^{1/2}$

$$U_b = v_\theta\Big|_{\text{at } r = (\chi/U_0)^{1/2}} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}\Big|_{\text{at } r = (\chi/U_0)^{1/2}} \tag{7.50}$$

$$= -2U_0 \sin\theta$$

From Eqs (7.49) and (7.50) we obtain

$$p_b = \rho g\left[\frac{U_0^2}{2g} + \frac{p_0}{\rho g} - \frac{(2U_0 \sin\theta)^2}{2g}\right] \tag{7.51}$$

The drag is calculated by integrating the force components arising out of pressure, in the x direction on the boundary. Referring to Fig. 7.12, the drag force can be written as

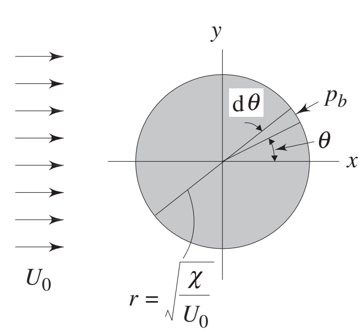

Fig 7.12 Calculation of drag on a cylinder

$$D = -\int_0^{2\pi} p_b \cos\theta \left(\frac{\chi}{U_0}\right)^{1/2} d\theta$$

or
$$D = -\int_0^{2\pi} \rho g \left(\frac{\chi}{U_0}\right)^{1/2} \left[\frac{U_0^2}{2g} + \frac{p_0}{\rho g} - \frac{(2U_0 \sin\theta)^2}{2g}\right] \cos\theta \, d\theta$$

$$D = -\int_0^{2\pi} \left[p_0 + \frac{\rho U_0^2}{2}(1 - 4\sin^2\theta)\right]\left(\frac{\chi}{U_0}\right)^{1/2} \cos\theta \, d\theta \qquad (7.52)$$

Similarly, the lift force

$$L = -\int_0^{2\pi} p_b \sin\theta \left(\frac{\chi}{U_0}\right)^{1/2} d\theta \qquad (7.53)$$

The Eqs (7.52) and (7.53) produce $D = 0$ and $L = 0$ after the integration is carried out.

However, in reality, the cylinder will always experience some drag force. This contradiction between the inviscid flow result and the experiment is usually known as *D'Almbert paradox*. The reason for the discrepancy lies in completely ignoring the viscous effects throughout the flow field. Effect of the thin region adjacent to the solid boundary is of paramount importance in determining drag force. However, the lift may often be predicted by the present technique. We shall appreciate this fact in a subsequent section.

7.3.4 Flow About a Rotating Cylinder

In addition to superimposed uniform flow and a doublet, a vortex is thrown at the doublet centre. This will simulate a rotating cylinder in uniform stream. We shall see that the pressure distribution will result in a force, a component of which will culminate in lift force. The phenomenon of generation of lift by a rotating object placed in a stream is known as *Magnus effect*. The velocity potential and stream functions for the combination of doublet, vortex and uniform flow are

$$\phi = U_0\, x + \frac{\chi\cos\theta}{r} - \frac{\Gamma}{2\pi}\theta \text{ (clockwise rotation)} \qquad (7.54)$$

$$\psi = U_0\, y - \frac{\chi\sin\theta}{r} + \frac{\Gamma}{2\pi} \ln r \text{ (clockwise rotation)} \qquad (7.55)$$

By making use of either the stream function or velocity potential function, the velocity components are

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \left(U_0 - \frac{\chi}{r^2}\right)\cos\theta \qquad (7.56)$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\left(U_0 + \frac{\chi}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r} \qquad (7.57)$$

Implicit in the above derivation are $x = r\cos\theta$ and $y = r\sin\theta$. At the stagnation points the velocity components must vanish. From Eq. (7.56), we get

$$\cos\theta\left(U_0 - \frac{\chi}{r^2}\right) = 0 \tag{7.58}$$

From Eq. (7.58) it is evident that a zero radial velocity component may occur at $\theta = \pm\dfrac{\pi}{2}$ and along the circle, $r = \left(\dfrac{\chi}{U_0}\right)^{1/2}$. Eq. (7.57) depicts that a zero transverse velocity requires

$$\sin\theta = \frac{-\Gamma/2\pi r}{U_0 + (\chi/r^2)} \quad \text{or} \quad \theta = \sin^{-1}\left[\frac{-\Gamma/2\pi r}{U_0 + \dfrac{\chi}{r^2}}\right] \tag{7.59}$$

However, at the stagnation point, both radial and transverse velocity components must be zero.
So, the location of stragnation point occurs at

$$r = \left(\frac{\chi}{U_0}\right)^{1/2}$$

and

$$\theta = \sin^{-1}\frac{\left\{-\Gamma\Big/\left(2\pi\left(\dfrac{\chi}{U_0}\right)^{1/2}\right)\right\}}{\left[U_0 + \chi\Big/\left(\dfrac{\chi}{U_0}\right)\right]}$$

or

$$\theta = \sin^{-1}\left[\frac{-\Gamma}{2\pi\left(\dfrac{\chi}{U_0}\right)^{1/2}}\cdot\frac{1}{2U_0}\right]$$

or

$$\theta = \sin^{-1}\left[\frac{-\Gamma}{4\pi(\chi U_0)^{1/2}}\right] \tag{7.60}$$

There will be two stagnation points since there are two angles for a given sine except for $\sin^{-1}(\pm 1)$.

The streamline passing through these points may be determined by evaluating ψ at these points. Substitution of the stagnation coordinate (r, θ) into the stream function (Eq. 7.55) yields

$$\psi = \left[U_0\left(\frac{\chi}{U_0}\right)^{1/2} - \frac{\chi}{\left(\dfrac{\chi}{U_0}\right)^{1/2}}\right]\sin\sin^{-1}\left[\frac{-\Gamma}{4\pi(\chi U_0)^{1/2}}\right] + \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2}$$

$$\psi = \left[(U_0\chi)^{1/2} - (U_0\chi)^{1/2}\right]\left[\frac{-\Gamma}{4\pi(\chi U_0)^{1/2}}\right] + \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2}$$

or $$\psi_{\text{stag}} = \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2} \tag{7.61}$$

Equating the general expression for stream function to the above constant, we get

$$U_0 r\sin\theta - \frac{\chi\sin\theta}{r} + \frac{\Gamma}{2\pi}\ln r = \frac{\Gamma}{2\pi}\ln\left(\frac{\chi}{U_0}\right)^{1/2}$$

By rearranging we can write

$$\sin\theta\left[U_0 r - \frac{\chi}{r}\right] + \frac{\Gamma}{2\pi}\left[\ln r \pm \ln\left(\frac{\chi}{U_0}\right)^{1/2}\right] = 0 \tag{7.62}$$

All points along the circle $r = \left(\frac{\chi}{U_0}\right)^{1/2}$ satisfy Eq. (7.62), since for this value of r, each quantity within parentheses in the equation is zero. Considering the interior of the circle (on which $\psi = 0$) to be a solid cylinder, the outer streamline pattern is shown in Fig. 7.13.

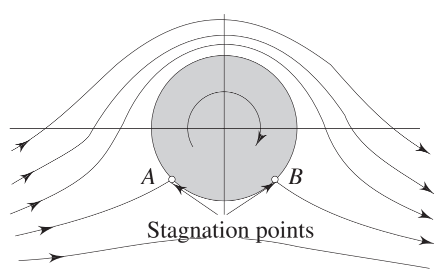

Fig. 7.13 Flow past a cylinder with circulation

A further look into Eq. (7.60) explains that at the stagnation point

$$\theta = \sin^{-1}\left[\frac{-(\Gamma/2\pi)}{2(\chi U_0)^{1/2}}\right]$$

or $$\theta = \sin^{-1}\left[\frac{-(\Gamma/2\pi)}{2U_0 r}\right] \tag{7.63}$$

The limiting case arises for $\frac{(\Gamma/2\pi)}{U_0 r} = 2$, where $\theta = \sin^{-1}(-1) = -90°$ and two stagnation points meet at the bottom as shown in Fig. 7.14.

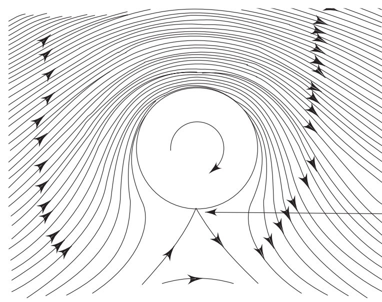

Fig. 7.14 Flow past a circular cylinder with circulation value $\frac{\Gamma/2\pi}{U_o r} = 2$

However, in all these cases the effects of the vortex and doublet become negligibly small as one moves a large distance from the cylinder. The flow is assumed to be uniform at infinity. We have already seen that the change in strength Γ of the vortex changes the flow pattern, particularly the position of the stagnation points but the radius of the cylinder remains unchanged.

7.3.5 Lift and Drag for Flow About a Rotating Cylinder

The pressure at large distances from the cylinder is uniform and given by p_0. Deploying Bernoulli's equation between the points at infinity and on the boundary of the cylinder,

$$p_b = \rho g\left[\frac{U_0^2}{2g} + \frac{p_0}{\rho g} - \frac{U_b^2}{2g}\right] \tag{7.64}$$

The velocity U_b is as such $v_\theta\Big|_{r=\left(\frac{\chi}{U_0}\right)^{1/2}}$

Hence, $$U_b = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -2U_0 \sin\theta - \frac{\Gamma}{2\pi}\left[\frac{U_0}{\chi}\right]^{1/2} \tag{7.65}$$

From Eqs (7.64) and (7.65) we can write

$$p_b = \rho g\left[\frac{U_0^2}{2g} + \frac{p_0}{\rho g} - \frac{\left[-2U_0\sin\theta - \frac{\Gamma}{2\pi}\left(\frac{U_0}{\chi}\right)^{\frac{1}{2}}\right]^2}{2g}\right] \tag{7.66}$$

The lift may calculated as (refer Fig. 7.12)

$$L = -\int_0^{2\pi} p_b \sin\theta \left[\frac{\chi}{U_0}\right]^{\frac{1}{2}} d\theta$$

or
$$L = -\int_0^{2\pi} \left\{ \frac{\rho U_0^2}{2} + p_0 - \frac{\rho\left[-2U_0 \sin\theta - \frac{\Gamma}{2\pi}\left(\frac{U_0}{\chi}\right)^{\frac{1}{2}}\right]^2}{2} \right\} \left[\frac{\chi}{U_0}\right]^{\frac{1}{2}} (\sin\theta)\, \mathrm{d}\theta$$

or
$$L = -\int_0^{2\pi} \left[\frac{\rho U_0^2}{2}\left(\frac{\chi}{U_0}\right)^{\frac{1}{2}} \sin\theta + p_0 \left(\frac{\chi}{U_0}\right)^{\frac{1}{2}} \sin\theta - \frac{\rho}{2}\left\{4U_o^2 \sin^2\theta + \frac{4U_0\Gamma\sin\theta}{2\pi}\left(\frac{U_0}{\chi}\right)^{\frac{1}{2}} + \frac{\Gamma^2}{4\pi^2}\left[\frac{U_0}{\chi}\right]\right\}\left(\frac{\chi}{U_0}\right)^{\frac{1}{2}} \sin\theta \right] \mathrm{d}\theta$$

or
$$L = -\int_0^{2\pi} \left[\frac{\rho U_0^2}{2}\left(\frac{\chi}{U_0}\right)^{\frac{1}{2}} \sin\theta + p_0 \left(\frac{\chi}{U_0}\right)^{\frac{1}{2}} \sin\theta - 2\rho U_0^2 \sin^3\theta \left(\frac{\chi}{U_o}\right)^{\frac{1}{2}} - \frac{\rho U_0 \Gamma}{\pi}\sin^2\theta - \frac{\rho\Gamma^2}{8\pi^2}\left(\frac{U_0}{\chi}\right)^{\frac{1}{2}} \sin\theta \right] \mathrm{d}\theta$$

or
$$L = \rho U_0 \Gamma \tag{7.67}$$

The drag force, which includes the multiplication by cos θ (and integration over 2π) is zero.

Thus the inviscid flow also demonstrates lift. It can be seen that the lift becomes a simple formula involving only the density of the medium, free stream velocity and circulation. In addition, it can also be shown that in two dimensional incompressible steady flow about a boundary of any shape, the lift is always a product of these three quantities.

The validity of Eq. (7.67) for any two-dimensional incompressible steady potential flow around a body of any shape, not necessarily a circular cylinder, is known as the *Kutta-Joukowski theorem* named after the German fluid dynamist Wilhelm Kutta (1867–1944) and Russian mathematician Nikolai J. Joukowski (1847–1921). A very popular example of the lift force acting on a rotating body is observed in the game of soccer. If a player imparts rotation on the ball while shooting it, instead of following the usual trajectory, the ball will swerve in the air and puzzle the goalkeeper. The swerve in the air can be controlled by varying the strength of circulation, i.e., the amount of rotation. In 1924, a man named *Flettner* had a ship built in Germany which possessed two rotating cylinders to generate thrust normal to wind blowing past the ship. The Flettner design did not gain any popularity but it is of considerable scientific interest (shown in Fig. 7.15).

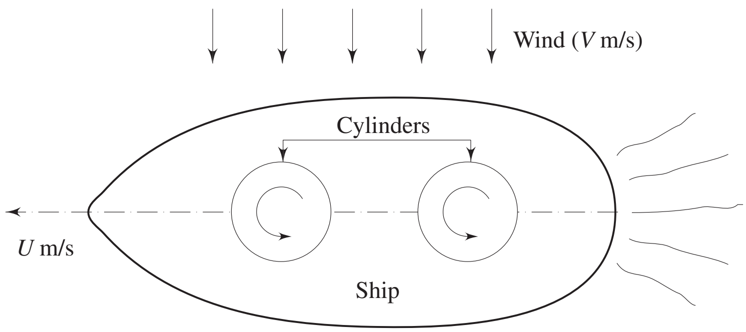

Fig. 7.15 Schematic diagram of the plan view of Flettner's ship

7.4 AEROFOIL THEORY

Aerofoils are streamline shaped wings which are used in airplanes and turbomachinery. These shapes are such that the drag force is a very small fraction of the lift. The following nomenclatures are used for defining an aerofoil (refer to Fig. 7.16).

The chord (c) is the distance between the leading edge and trailing edge. The length of an aerofoil, normal to the cross-section (i.e., normal to the plane of a paper) is called the span of aerofoil. The camber line represents the mean profile of the aerofoil. Some important geometrical parameters for an aerofoil are the ratio of maximum thickness to chord (t/c) and the ratio of maximum camber to chord (h/c). When these ratios are small, an aerofoil can be considered to be thin. For the analysis of flow, a thin aerofoil is represented by its camber.

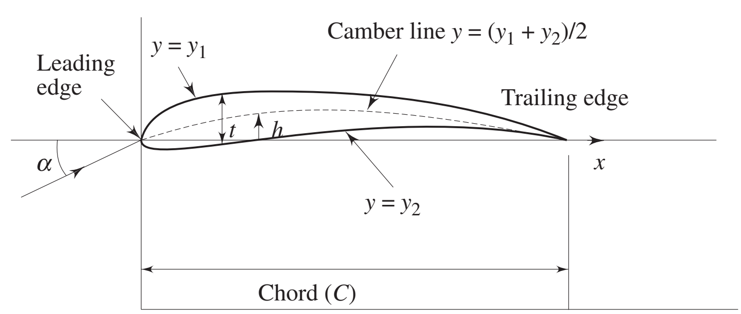

Fig. 7.16 Aerofoil section

The theory of thick cambered aerofoils is an advanced topic. Basically it uses a complex-variable mapping which transforms the inviscid flow across a rotating cylinder into the flow about an aerofoil shape with circulation.

7.4.1 Flow Around a Thin Aerofoil

Thin aerofoil theory is based upon the superposition of uniform flow at infinity and a continuous distribution of clockwise free vortex on the camber line having

circulation density $\gamma(s)$ per unit length. The circulation density $\gamma(s)$ should be such that the resultant flow is tangent to the camber line at every point.

Since the slope of the camber line is assumed to be small, $\gamma(s)\mathrm{d}s = \gamma(\eta)\mathrm{d}\eta$ (refer Fig. 7.17). The total circulation around the profile is given by

$$\Gamma = \int_0^C \gamma(\eta)\,\mathrm{d}\eta \tag{7.68}$$

A vortical motion of strength $\gamma\,\mathrm{d}\eta$ at $x = \eta$ develops a velocity at the point P which may be expressed as

$$\mathrm{d}v = \frac{\gamma(\eta)\,\mathrm{d}\eta}{2\pi(\eta - x)} \text{ acting upwards}$$

The total induced velocity in the upward direction at P due to the entire vortex distribution along the camber line is

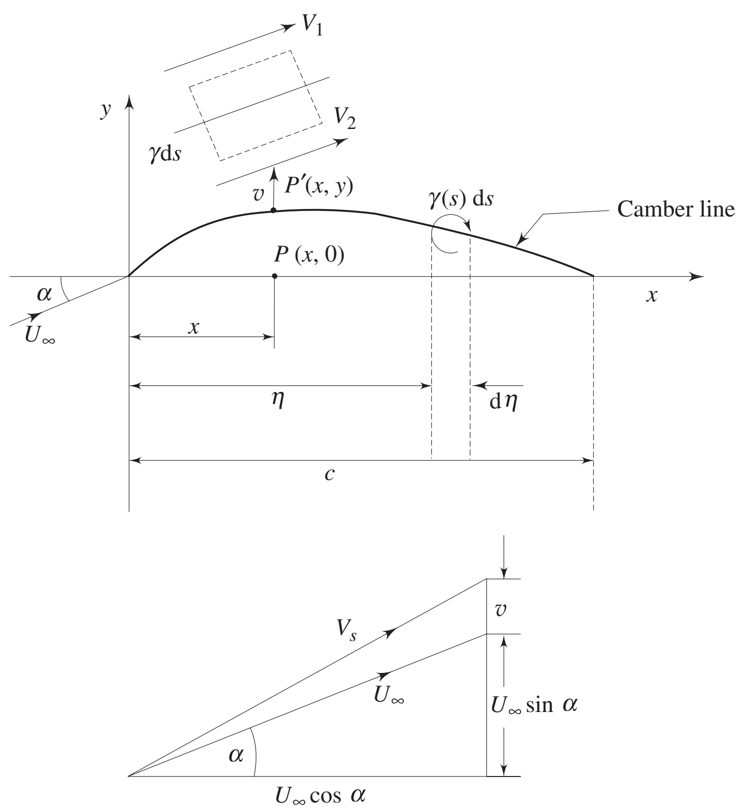

Fig. 7.17 Flow around thin aerofoil

$$v(x) = \frac{1}{2\pi}\int_0^C \frac{\gamma(\eta)\,\mathrm{d}\eta}{(\eta - x)} \tag{7.69}$$

For a small camber (having small α), this expression is identically valid for the induced velocity at P' due to the vortex sheet of variable strength $\gamma(s)$ on the camber line. The resultant velocity due to U_∞ and $v(x)$ must be tangential to the camber line so that the slope of a camber line may be expressed as

$$\frac{dy}{dx} = \frac{U_\infty \sin\alpha + v}{U_\infty \cos\alpha} = \tan\alpha + \frac{v}{U_\infty \cos\alpha}$$

or

$$\frac{dy}{dx} = \alpha + \frac{v}{U_\infty} \quad \text{[since } \alpha \text{ is very small]} \tag{7.70}$$

From Eqs (7.69) and (7.70) we can write

$$\frac{dy}{dx} = \alpha + \frac{1}{2\pi U_\infty}\int_0^C \frac{\gamma(\eta)\,d\eta}{(\eta - x)} \tag{7.71*}$$

Let us consider an element ds on the camber line. Consider a small rectangle (drawn with dotted line) around ds. The upper and lower sides of the rectangle are very close to each other and these are parallel to the camber line. The other two sides are normal to the camber line. The circulation along the rectangle is measured in clockwise direction as

$$V_1\,ds - V_2\,ds = \gamma\,ds \quad \text{[normal component of velocity at the camber line should be zero]}$$

or

$$V_1 - V_2 = \gamma \tag{7.72}$$

If the mean velocity in the tangential direction at the camber line is given by $V_s = (V_1 + V_2)/2$, it can be rewritten as

$$V_1 = V_s + \frac{\gamma}{2} \quad \text{and} \quad V_2 = V_s - \frac{\gamma}{2}$$

In the event, it can be said that if v is very small [$v << U_\infty$], V_s becomes equal to U_∞. The difference in velocity across the camber line brought about by the vortex sheet of variable strength $\gamma(s)$ causes pressure difference and generates lift force.

7.4.2 Generation of Vortices Around a Wing

The lift around an aerofoil is generated following *Kutta-Joukowski theorem*. Lift is a product of ρ, U_∞ and the circulation Γ. Mechanism of induction of circulation is to be understood clearly.

When the motion of a wing starts from rest, vortices are formed at the trailing edge (refer Fig. 7.18).

At the start, there is a velocity discontinuity at the trailing edge. This is eventual because near the trailing edge, the velocity at the bottom surface is

* For a given aerofoil, the left hand side term of the integral Eq. (7.71) is a known function. Finding out $\gamma(\eta)$ from it is a formidable task. This exercise is not being discussed in this text. Interested readers may refer to the books by Glauert [1] and Batchelor [2]. If $\gamma(\eta)$ is determined, the circulation Γ and consequently the lift $L = \rho U_\infty \Gamma$ can easily be calculated.

higher than that at the top surface. This discrepancy in velocity culminates in the formation of vortices at the trailing edge. Figure 7.18 (a) depicts the formation of starting vortex by impulsively moving aerofoil. However, the starting vortices induce a counter circulation as shown in Figure 7.18 (b). The circulation around a path (*ABCD*) enclosing the wing and just shed (starting) vortex must be zero. Here we refer to Kelvin's theorem once again.

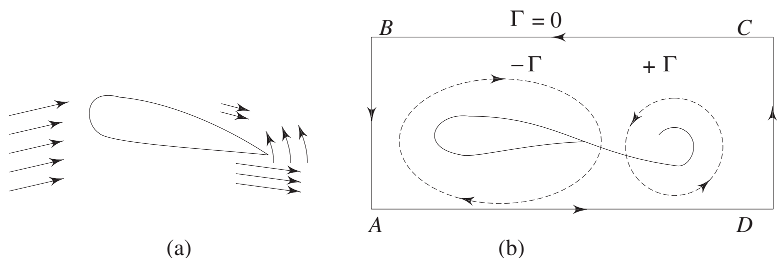

Fig 7.18 Vortices generated when an aerofoil just begins to move

Initially the flow starts with the zero circulation around the closed path. Thereafter, due to the change in angle of attack or flow velocity, if a fresh starting vortex is shed, the circulation around the wing will adjust itself so that a net zero vorticity is set around the closed path.

The discussions in the previous section were for two-dimensional, infinite span wings. But real wings have finite span or finite aspect ratio λ, defined as

$$\lambda = \frac{b^2}{A_s} \tag{7.73}$$

where b is the span length and A_s is the plan form area as seen from the top. For a wing of finite span, the end conditions affect both the lift and the drag. In the leading edge region, pressure at the bottom surface of a wing is higher than that at the top surface. The longitudinal vortices are generated at the edges of finite wing owing to pressure differences between the bottom surface directly facing the flow and the top surface (refer Fig. 7.19). This is very prominent for small aspect ratio delta wings which are used in high-performance aircrafts as shown in Fig. 7.20.

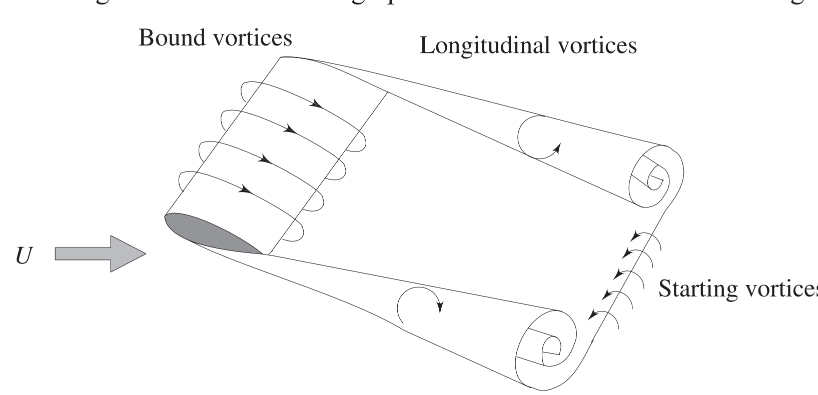

Fig. 7.19 Vortices around a finite wing

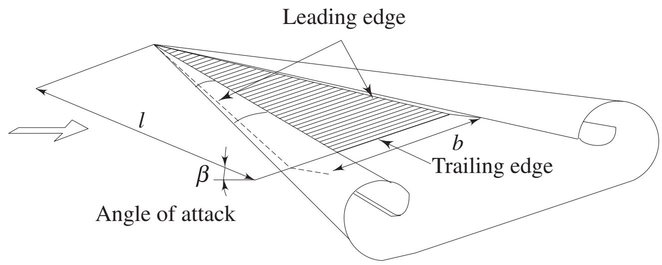

Fig. 7.20 Counter rotating leading edge vortices generated by a delta-wing

However, circulation around a wing gives rise to bound vortices that move along with the wing. In 1918, *Prandtl* successfully modelled such flows by replacing the wing with a lifting line. The bound vortices around this lifting line, the starting vortices and the longitudinal vortices formed at the edges, constitute a closed vortex ring as shown in Fig. 7.19.

Summary

This chapter has given a brief description of inviscid, incompressible, irrotational flows.

- Irrotationality leads to the condition $\nabla \times \vec{V} = 0$ which demands $\vec{V} = \nabla\phi$, where ϕ is known as a potential function. For a potential flow $\nabla^2\phi = 0$.
- The stream function ψ also obeys the Laplace's equation $\nabla^2 \psi = 0$ for the potential flows. Laplace's equation is linear, hence any number of particular solutions of Laplace's equation added together will yield another solution. So a complicated flow for an inviscid, incompressible, irrotational condition can be synthesized by adding together a number of elementary flows which are also inviscid, incompressible and irrotational. This is called the method of superposition.
- Some inviscid flow configurations of practical importance are solved by using the method of superposition. The circulation in a flow field is defined as $\Gamma = \int \vec{V} \cdot d\vec{s}$. Subsequently, the vorticity may be defined as circulation per unit area. The circulation for a closed path in an irrotational flow field is zero. However, the circulation for a given closed path in an irrotational flow containing a finite number of singular points is a non-zero constant.
- The lift around an immersed body is generated when the flow field possesses circulation. The lift around a body of any shape is given by $L = \rho U_0 \Gamma$, where ρ is the density and U_0 is the velocity in the streamwise direction.

Solved Examples

Example 7.1 The velocity components of two dimensional incompressible flow are $u = 2xy$ and $v = a^2 + x^2 - y^2$. Show that a velocity potential function exists and find out the velocity potential.

Solution The velocity potential function exists only for irrotational flow. The condition to be satisfied is

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Evaluating the derivatives mentioned above, we get

$$\frac{\partial v}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 2x$$

The flow is irrotational. From definition we can also write

$$u = \frac{\partial \phi}{\partial x}$$

$$\frac{\partial \phi}{\partial x} = 2\,xy \qquad \text{or} \qquad \phi = x^2 y + f_1\,(y)$$

also, $$v = \frac{\partial \phi}{\partial y}$$

or $$\frac{\partial \phi}{\partial y} = a^2 + x^2 - y^2 \qquad \text{or} \qquad \phi = a^2 y + x^2 y - \frac{y^3}{3} + f_2(x)$$

Since both the solutions are same, we can write

$$x^2 y + f_1(y) = a^2 y + x^2 y - \frac{y^3}{3} + f_2(x)$$

or $$f_1(y) = a^2 y - \frac{y^3}{3} + f_2(x)$$

In order to keep the above expression valid for all the values of y, $f_2(x)$ has to be a constant.

Thus $$\phi = a^2 y + x^2 y - \frac{y^3}{3} + \text{constant}$$

Since ϕ = constant and represents a family of lines, ϕ may be written without a constant as

$$\phi = a^2 y + x^2 y - \frac{y^3}{3}$$

Example 7.2 The flow of an incompressible fluid is defined by $u = 2$, $v = 8x$. Does a stream function exist? If so, find its expression.

Solution Compliance of continuity describes the existence of a stream function

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial (2)}{\partial x} + \frac{\partial}{\partial y}(8x) = 0$$

So, the stream function exists.
Now we can write

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\,\mathrm{d}x + \frac{\partial \psi}{\partial y}\,\mathrm{d}y$$

or $$\mathrm{d}\psi = -\,v\mathrm{d}x + u\mathrm{d}y$$

or $\quad d\psi = -8x\,dx + 2dy$

or $\quad \psi = -4x^2 + 2y + C$

Dropping the constant C, $\quad \psi = -4x^2 + 2y.$

Example 7.3 Does a velocity potential function $\phi = 2(x^2 + 2y - y^2)$ describe the possible flow of an incompressible fluid? If so, find out the equation for the velocity vector $\vec{\mathbf{V}}$. Also determine the equation for streamlines.

Solution For the given ϕ, in order to describe an incompressible flow, we check with the Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2(2) + 2(-2) = 0$$

So, a flow field exists.

The velocity components are

$$u = \frac{\partial\phi}{\partial x} = 2(2x) = 4x$$

$$v = \frac{\partial\phi}{\partial y} = 2\,(2 - 2y) = 4 - 4y$$

Velocity vector $\quad \vec{\mathbf{V}} = 4x\,\hat{\mathbf{i}} + (4 - 4y)\,\hat{\mathbf{j}}$

Stream function ψ can be expressed as

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$$

or $\quad d\psi = -v\,dx + u\,dy$

or $\quad d\psi = -(4 - 4y)\,dx + 4x\,dy$

$$\psi = -\int (4 - 4y)dx + \int 4x\,dy + C$$

$$\psi = -4x + 4xy + 4xy + C$$

Dropping the constant C, stream function becomes

$$\psi = 4\,(2xy - x)$$

Example 7.4 The radial velocity of a flow is described by $v_r = \dfrac{k}{\sqrt{r}}\cos\theta$. If $v_\theta = 0$ at $\theta = 0$, find out v_θ and the stream function for the flow.

Solution $\quad v_r = \dfrac{1}{r}\cdot\dfrac{\partial\psi}{\partial\theta} = \dfrac{k}{\sqrt{r}}\cos\theta$

or $\quad \dfrac{\partial\psi}{\partial\theta} = k\sqrt{r}\cos\theta$

or $\quad \psi = k\sqrt{r}\sin\theta = f(r)$

Now $\quad v_\theta = -\dfrac{\partial\psi}{\partial r} = \dfrac{k}{2\sqrt{r}}\sin\theta + f'(r)$

We know, $\quad v_\theta = 0$ at $\theta = 0$, which depicts

$f'(r) = 0 \quad$ and $\quad f(r) = $ constant

Therefore $$v_\theta = \frac{k}{2\sqrt{r}} \sin\theta \text{ and}$$

$$\psi = k\sqrt{r}\sin\theta$$

Example 7.5 A two dimensional source of volume flow rate $\Lambda = 2.5\ \text{m}^2/\text{s}$ is located in a uniform flow (U_0) of 2 m/s. Determine the stagnation point and the maximum thickness of resulting half body.

Solution We have already constructed different flow patterns by superimposing elementary flows. An interesting body shape appears if we superimpose a uniform flow over an isolated source or sink which is known as Rankine half body (refer Fig. 7.21). Let the source be located at the origin.

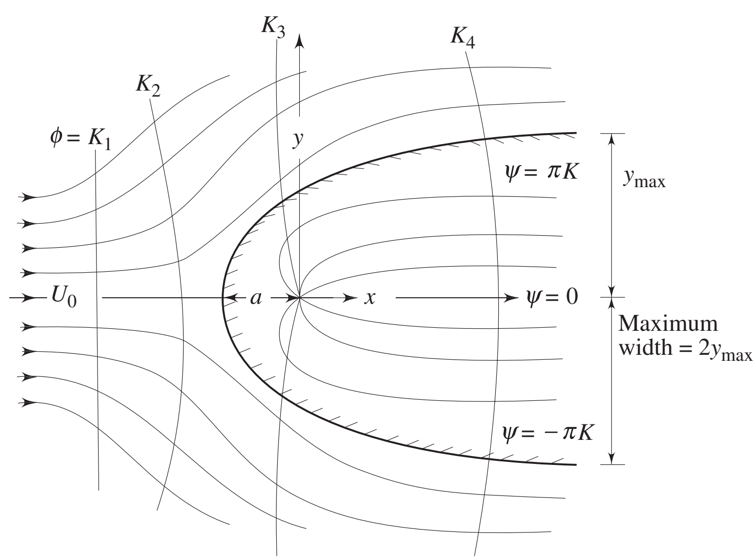

Fig. 7.21 Uniform flow plus source equals a half body

(i) Then the stream function of combination is

$$\psi = U_0 y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

or $$\psi = U_0 y + K\tan^{-1}\left(\frac{y}{x}\right)$$

Velocity $$u = \frac{\partial\psi}{\partial y} = U_0 + K\frac{x}{x^2+y^2}$$

Similarly $$v = -\frac{\partial\psi}{\partial x} = +K\frac{y}{x^2+y^2}$$

At the stagnation point $u = 0$, $v = 0$, demand from the above equation $y = 0$ and

$$x = -\frac{\Lambda}{2\pi U_0} = -\frac{2.5}{2\pi\times 2} = -0.2\ \text{m}$$

The coordinates of stagnation points are $(-a, 0)$ or $(-0.2, 0)$

The value of stream function at the stagnation point is

$$\psi(-0.2, 0) = 0 + \frac{\Lambda}{2\pi}\tan^{-1} 0$$

or $$\psi_{stag} = 0 + \frac{2.5}{2\pi}\pi = 1.25 \text{ m}^2/\text{s}$$

The half-body is described by dividing streamline

$$\psi = \frac{\Lambda}{2} = \pi \cdot \frac{\Lambda}{2\pi} = \pi K$$

or $$U_0 y + \frac{\Lambda}{2\pi}\tan^{-1}\frac{y}{x} = \frac{\Lambda}{2}$$

or $$U_0 y + \frac{\Lambda\theta}{2\pi} = \frac{\Lambda}{2} \quad \text{or} \quad y = \frac{\Lambda\left(1 - \frac{\theta}{\pi}\right)}{2\,U_0}$$

at $\theta = 0$ $\quad y_{max} = \dfrac{\Lambda}{2U_0}$, the maximum ordinate

at $\theta = \dfrac{\pi}{2}$, $\quad y = \dfrac{\Lambda}{4U_0}$, the upper ordinate at the origin

at $\theta = \pi$, $\quad y = 0$, the stagnation point

at $\theta = \dfrac{3\pi}{2}$, $\quad y = -\dfrac{\Lambda}{4U_0}$, the lower ordinate at the origin

(ii) However, the equation of the half body becomes

$$U_0 y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) = \frac{\Lambda}{2}$$

The maximum thickness occurs as $x \to \infty$

$$2y + \frac{1.25}{\pi}\tan^{-1} 0 = 1.25$$

$$y = \frac{1.25}{2} = 0.625 \text{ m}$$

The maximum thickness $= 2y_{max} = 1.25$ m

Example 7.6 A source at the origin and a uniform flow at 5m/s are superimposed. The half-body which is formed has a maximum width of 2 m. Calculate (i) location of stagnation point (ii) width of the body at the origin and (iii) velocity at a point $\left(0.7, \frac{\pi}{2}\right)$.

Solution (i) We have seen in Example 7.5, that

at $\theta = 0$, $\quad y_{max} = \dfrac{\Lambda}{2U_0} = \dfrac{2}{2} = 1$ m

or $\Lambda = 2 \times 5 \times 1 = 10 \text{ m}^2/\text{s}$

for stagnation point,

$$x = -\frac{K}{U_0} = -\frac{\Lambda}{2\pi U_0} = \frac{-10}{2\pi \times 5} = -0.32 \text{ m}$$

and $y = 0$

(ii) at $\theta = \frac{\pi}{2}$, $y = \frac{\Lambda}{4U_0}$ [from Example 7.5]

at $\theta = \frac{\pi}{2}$, $y = \frac{10}{4 \times 5} = 0.5$ m

The width of the body at the origin is $2 \times 0.5 = 1$ m

(iii) In polar coordinate

$$\psi = U_0 r \sin\theta + K\theta \quad \text{where} \quad K = \frac{\Lambda}{2\pi}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_0 \cos\theta + \frac{\Lambda}{2\pi r}$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U_0 \sin\theta$$

at the point $(0.7, \pi/2)$

$$v_r = \frac{\Lambda}{2\pi r} = \frac{10}{2\pi \times 0.7} = 2.27 \text{ m/s}$$

$$v_\theta = -U_0 \sin\theta = -5 \sin\frac{\pi}{2} = -5 \text{ m/s}$$

$$V_{\text{resultant}} = \sqrt{(2.27)^2 + (5)^2} = 5.49 \text{ m/s}$$

Example 7.7 A line source discharging a flow at 0.6 m^2/s per unit length is located at (–1,0) and a sink of volume flow rate 1.2 m^2/s is located at (2,0). For a dynamic pressure of 10 N/m^2 at the origin, determine the velocity and dynamic pressure at (1,1).

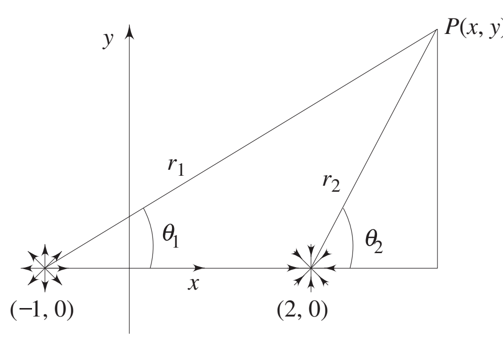

Fig. 7.22 Source and sink pair

Solution ψ at P may be expressed as

$$\psi = K_1\theta_1 - K_2\theta_2$$

or
$$\psi = \frac{\Lambda_1}{2\pi}\theta_1 - \frac{\Lambda_2}{2\pi}\theta_2$$

or
$$\psi = \frac{0.6}{2\pi}\tan^{-1}\left(\frac{y}{x+1}\right) - \frac{1.2}{2\pi}\tan^{-1}\left(\frac{y}{x-2}\right)$$

$$u = \frac{\partial \psi}{\partial y} = \frac{0.6}{2\pi}\left[\frac{(x+1)}{(x+1)^2 + y^2}\right] - \frac{1.2}{2\pi}\left[\frac{(x-2)}{(x-2)^2 + y^2}\right]$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{0.6}{2\pi}\left[\frac{y}{(x+1)^2 + y^2}\right] - \frac{1.2}{2\pi}\left[\frac{y}{(x-2)^2 + y^2}\right]$$

at the origin (0,0)

$$u = \frac{0.6}{2\pi} - \frac{1.2}{2\pi}\left(\frac{-2}{4}\right) = \frac{0.6}{2\pi} + \frac{0.6}{2\pi} = \frac{0.6}{\pi} = 0.1909 \text{ m/s}$$

$$v = 0$$

Dynamic pressure $\frac{1}{2}\rho V^2 = 10 \text{ N/m}^2$

or $$\rho = \frac{20}{V^2} = 548.3 \text{ kg/m}^3$$

At point (1,1)

$$u = \frac{0.6}{2\pi}\cdot\frac{2}{5} + \frac{1.2}{2\pi}\cdot\frac{1}{2} = \frac{0.6}{5\pi} + \frac{0.6}{2\pi}$$

$$= 0.0381 + 0.0954 = 0.1335 \text{ m/s}$$

$$v = \frac{0.6}{2\pi}\cdot\frac{1}{5} - \frac{1.2}{2\pi}\cdot\frac{1}{2} = \frac{0.6}{10\pi} + \frac{0.6}{2\pi}$$

$$= 0.019 - 0.0954 = -0.0764 \text{ m/s}$$

$$V_{\text{resultant}} = 0.1538 \text{ m/s}$$

Dynamic pressure at (1,1) $= \frac{1}{2} \times 548.3 \times (0.1538)^2$

$$= 6.48 \text{ N/m}^2$$

Example 7.8 The wind velocity at a location 5 km away from the centre of a tornado (consider inviscid, irrotational vortex motion) was measured as 30 km/hr and the barometric pressure was 750 mm of Hg. Calculate the wind velocity 1 km from the tornado centre and its barometric pressure. [Density of air = 1.2 kg/m^3, density of mercury = 13.6 × 10^3 kg/m^3].

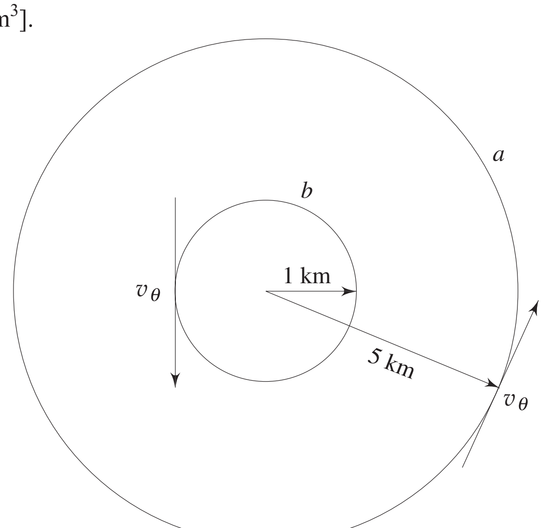

Fig. 7.23 Model of a tornado (irrotational vortex)

Solution $p_a = 750$ mm of Hg $= 0.75$ m of Hg

$$= 0.75 \times 13.6 \times 10^3 \times 9.81 = 100.062 \text{ kN/m}^2$$

From free vortex consideration, we can write

$$r_a\, v_{\theta a} = r_b\, v_{\theta b} = C$$

$$r_a = 5000 \text{ m}, v_{\theta a} = \frac{30 \times 1000}{60 \times 60} = 8.33 \text{ m/s}$$

$$C = \text{circulation constant} = 41650 \text{ m}^2/\text{s}$$

at $\quad r_b = 1000$ m,

$$v_{\theta b} = \frac{41650}{1000} = 41.65 \text{ m/s}$$

Bernoulli's equation between points a and b

$$\frac{p_a}{\rho g} + \frac{V_{\theta a}^2}{2g} = \frac{p_b}{\rho g} + \frac{V_{\theta b}^2}{2g}$$

$$\frac{100062}{1.2 \times 9.81} + \frac{(8.33)^2}{2 \times 9.81} = \frac{p_b}{\rho g} + \frac{(41.65)^2}{2 \times 9.81}$$

or $$\frac{p_b}{\rho g} = 8500 + 3.536 - 88.416$$

or $$\frac{p_b}{\rho g} = 8415.12$$

or $$p_b = 99062 \text{ N/m}^2 = 99.062 \text{ kN/m}^2$$

Example 7.9 A source with volume flow rate 0.2 m^2/s and a vortex with strength 1 m^2/s are located at the origin. Determine the equations for velocity potential and stream function. What should be the resultant velocity at $x = 0.9$ m and $y = 0.8$ m?

Solution

For the source $\quad \psi = K_1 \theta, \quad \phi = K_1 \ln r$

For the vortex $\quad \psi = -K_2 \ln r, \quad \phi = K_2 \theta$

Combined $$\psi = \frac{0.2}{2\pi}\theta - \frac{1}{2\pi}\ln r = \frac{1}{\pi}\left[0.1\,\theta - \frac{1}{2}\ln r\right]$$

Combined $$\phi = \frac{0.2}{2\pi}\ln r + \frac{1}{2\pi}\theta = \frac{1}{\pi}\left[0.1\ln r + \frac{1}{2}\theta\right]$$

Now $$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{10\pi r}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{2\pi r}$$

at $\quad x = 0.9$ m and $y = 0.8$ m

$$r = \sqrt{(0.9)^2 + (0.8)^2} = 1.204 \text{ m}$$

$$v_r(0.9, 0.8) = \frac{1}{10 \times \pi \times 1.204} = 0.026 \text{ m/s}$$

$$v_\theta(0.9, 0.8) = \frac{1}{2\,\pi r} = \frac{1}{2 \times \pi \times 1.204}$$

$$= 0.132 \text{ m/s}$$

$$V_{\text{resultant}} = \sqrt{(0.026)^2 + (0.132)^2} = 0.134 \text{ m/s}$$

Example 7.10 A 300 mm diameter circular cylinder is rotated about its axis in a stream of water having a uniform velocity of 5 m/s. Estimate the rotational speed when both the stagnation points coincide. Estimate the lift force experienced by the cylinder under such condition. ρ of water may be assumed to be 1000 kg/m^3.

Solution Stagnation point is given by

$$\theta = \sin^{-1}\left(\frac{-\Gamma}{4\pi r U_0}\right)$$

When both the stagnation points coincide, the two angles are equal and $\theta = -90°$. Stagnation point is at the lower surface [Fig. 7.14].

Thus $$\frac{\Gamma}{4\pi r U_0} = 1$$

or $$\Gamma = 4\pi r U_0$$

If the cylinder is rotating at an angular speed ω, the circulation due to the equivalent forced vortex is

$$\Gamma = \oint (\omega r)\, r\, d\theta = 2\pi r^2 \omega$$

$$2\pi \omega r^2 = 4\pi r U_0$$

$$\omega = \frac{2U_0}{r}$$

or $$\omega = \frac{2 \times 5}{0.15} = 66.67 \text{ rad/s}$$

and $$\Gamma = 4\pi \times 0.15 \times 5$$

$$= 9.42 \text{ m}^2/\text{s}$$

Lift force $$= L = \rho U \Gamma$$

or $$L = 1000 \times 5 \times 9.42 \left[\frac{\text{kg}}{\text{m}^3} \cdot \frac{\text{m}}{\text{s}} \cdot \frac{\text{m}^2}{\text{s}}\right]$$

or $$L = 47100 \text{ N/m}$$

As such, the lift is calculated per metre length of the cylinder

So $$\text{Lift} = 47.1 \text{ kN/m}^2$$

References

1. Glauert, H., *The Elements of Aerofoil and Airscrew Theory,* Cambridge University Press, London , 1926, 1948.
2. Batchelor, G.K., *An Introduction to Fluid Dynamics,* Cambridge University Press, London/New York, 1967, Reprinted 1970.

Exercises

7.1 Choose the correct answer for the following questions

(i) Stream function is defined for
   (a) all 3-D flow situations (b) flow of perfect fluid
   (c) irrotational flows only (d) 2-D incompressible flows.

(ii) Velocity potential exists for
   (a) all 3-D flow situations (b) flow of perfect fluid
   (c) all irrotational flows (d) steady irrotational flow

(iii) The continuity equation for a fluid states that
   (a) mass flow rate through a stream tube is constant.
   (b) the derivatives of velocity components exist at every point.
   (c) the velocity is tangential to stream lines.
   (d) the stream function exists for steady flows.

(iv) $\nabla . \vec{\mathbf{V}} = 0$ implies that
   (a) dilatation rate for a fluid particle is zero.
   (b) net mass flux from a control volume in any flow situation is zero.
   (c) the fluid is compressible.
   (d) density is a function of time only.

(v) Momentum theorem is valid only if the fluid is
   (a) incompressible (b) in irrotational motion
   (c) inviscid (d) irrespective of the above restrictions.

(vi) Circulation is defined as
   (a) line integral of velocity about any path
   (b) integral of tangential component of velocity about a path
   (c) line integral of velocity about a closed path
   (d) line integral of tangential component of velocity about a closed path.

(vii) In an irrotational flow, Stokes, theorem implies that circulation is zero
   (a) around two dimensional infinite bodies
   (b) in simply connected regions
   (c) in multiply connected regions
   (d) without any restriction.

(viii) The curl of a given velocity field indicates
   (a) the rate of increase or decrease of flow at a point
   (b) the rate of twisting of the lines of flow
   (c) the deformation rate
   (d) none of the above

7.2 Prove that the streamlines $\psi(r, \theta)$ in a polar coordinate system are orthogonal to the velocity potential lines $\phi(r, \theta)$.

7.3 The x and y components of velocity in a two-dimensional incompressible flow are given by

$$u = 3x + 3y \qquad \text{and} \qquad v = 2x - 3y$$

Derive an expression for the stream function. Show that the flow is rotational. Calculate the vorticity in the flow field.

Ans. $(-x^2 + 3xy + (3/2)\, y^2,\ \nabla \times \vec{\mathbf{V}} = -1)$

7.4 A source of volume flow rate 2 m^2/s is located at origin and another source of volume flow rate 4 m^2/s is located at (3,0). Find out the velocity components at (2, 2).

Ans. $u = -0.048$ m/s, $v = 0.334$ m/s)

7.5 A source of volume flow rate 5 m^2/s at the origin and a uniform flow of velocity 4 m/s combine to form a two-dimensional half body. Find out the maximum width of the half body. *Ans.* (1.25 m)

7.6 A source and a sink of equal volume flow rate 10 m^2/s are located 2 m apart. If a uniform flow of 5 m/s is superimposed, find out the location of the stagnation points.

Ans. (1.28, 0) and (– 1.28, 0)

7.7 The discharge of 30 m^2/s pollutants from a chemical plant into 10 m deep river, flowing at 0.3 m/s, can be modelled as a 2-D source across the river depth. It is found that the fishes in a certain zone die out whereas those outside the zone are unaffected. Find out the extent of this critical zone, if the point of discharge is in the midplane of a wide river.

Ans. (Rankine half body with stagnation point (15.91, 0) and $2y_{max} = 100$ m)

7.8 The Flettner rotor ship (Fig. 7.15) make use of two rotating vertical cylinders. Each has a diameter of 3 m and length of 15 m. If they rotate at a speed of 720 rpm, calculate the magnitude of Magnus force developed by the rotors in a breeze of 10 m/s. Assume air density as 1.22 kg/m^3.

Ans. (390.083 kN)

7.9 Find out the strength of a doublet which simulates a physical situation of 2 m diameter cylinder in a uniform flow of 15 m/s.

Ans. ($\Lambda = 47.124$ m^3/s per metre)

7.10 Consider a forced vortex rotated at an angular speed ω. Evaluate the circulation around any closed path in a forced vortex flow.

Derive the expression of hydrodynamic pressure as a function of radius for (i) a free vortex and (ii) a forced vortex.

Ans. (Forced vortex $\Gamma = 2\pi r^2 \omega$; (i) $p = -\rho C^2/r^2 + C_2$ (ii) $p = \rho\omega^2 r^2/2 + C_1$)

7.11 A tornado may be modelled as a circulating flow shown in Fig. 7.24 with $v_r = v_z = 0$

$$v_\theta = \omega r \text{ for } r \le R$$

$$= \frac{\omega R^2}{r} \text{ for } r \ge R$$

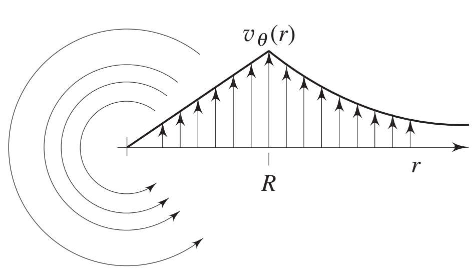

Fig. 7.24 Model of tornado (combination of free and forced vortex)

Determine whether the flow pattern is irrotational in either the inner ($r < R$) or the outer ($r > R$) region. Using r momentum equation, determine the pressure distribution $p(r)$ in the tornado, assuming $p = p_\infty$ at $r \to \infty$. Find out the location of the minimum pressure.

Ans. ($p = p_\infty - \rho\omega^2 R^2 (1 - r^2/2R^2)$ for $r < R$;
$p = p_\infty - \rho\omega^2 R^4 / 2r^2$ for $r > R$)

7.12 A 2 m diameter cylinder is rotating at 1400 rpm in an air stream flowing at 20 m/s. Calculate the lift and drag forces per unit depth of the cylinder. Assume air density as 1.22 kg/m^3. *Ans.* ($L = 22.476$ kN, $D = 0$)

7.13 Flow past a rotating cylinder can be simulated by superposition of a doublet, a uniform flow and a vortex. The peripheral velocity of the rotating cylinder alone is given by v_θ at $r = R$ (R is the radius of the cylinder). Use the expression for the combined velocity potential for the superimposed uniform flow, doublet and vortex flow (clockwise rotation) and show that the resultant velocity at any point on the cylinder is given by $-2U_o \sin\theta - v_\theta$ (at $r = R$). The angle θ is the angular position of the point of interest. A cylinder rotates at 360 rpm around its own axis which is perpendicular to the uniform air stream (density 1.24 kg/m^3) having a velocity of 25 m/s. The cylinder is 2 m in diameter. Find out (a) circulation, Γ (b) lift per unit length and the (c) position of the stagnation points.

Ans. (236.87 m^2/s, 7343 N/m, – 48.93° and 228.93°)

7.14 Flow past a rotating cylinder can be simulated by superposition of a doublet, a uniform flow and a vortex. The peripheral velocity of the rotating cylinder alone is given by v_θ at $r = R$ (R is the radius of the cylinder). Use the expression for the combined velocity potential for the superimposed uniform flow, doublet and vortex flow (clockwise rotation) and show that the resultant velocity at any point on the cylinder is given by $-2U_o \sin\theta - v_\theta$ (at $r = R$). The angle θ is the angular position of the point of interest. A cylinder rotates at 240 rpm around its own axis which is perpendicular to the uniform air stream (density 1.24 kg/m^3) having a velocity of 20 m/s. The cylinder is 2 m in diameter. Find out (a) circulation, Γ (b) lift per unit length and the (c) speed of rotation of the cylinder, which yields only a single stagnation point.

Ans. (157.91 m^2/s, 3916.25 N/m, 382 rpm)